

ON MONOTONE FOURIER COEFFICIENTS OF A FUNCTION
BELONGING TO NIKOL'SKIĬ-BESOV CLASSES

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ABSTRACT. In this paper, necessary and sufficient conditions on terms of monotone Fourier coefficients for a function to belong to a Nikol'skiĭ-Besov type class are given.

1. Let $f \in L_p[0, 2\pi]$, $1 < p < \infty$, be a 2π -periodic function having a cosine Fourier series with monotone coefficients, i.e.

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx, \quad a_n \downarrow 0.$$

and $\omega_k(f, t)_p$ the modulus of smoothness of order k in $L_p[0, 2\pi]$ metrics of the function f , i.e.

$$\omega_k(f, t)_p = \sup_{|h| \leq t} \|\Delta_h^k f\|_p,$$

where is

$$\Delta_h^k f(x) = \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu h).$$

We say that a 2π -periodic function f belongs to the Nikol'skiĭ-Besov class $N(p, \theta, r, \lambda, \varphi)$, $1 < p < \infty$, if the following conditions are satisfied

- (1) $f \in L_p[0, 2\pi]$;
- (2) Numbers θ, r, λ belong to the interval $(0, \infty)$, and k is an integer satisfying $k > r + \lambda$;
- (3) The following inequality holds true

$$\left(\int_0^\delta t^{-r\theta-1} \omega_k(f, t)_p^\theta dt + \delta^{\lambda\theta} \int_\delta^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^\theta dt \right)^{1/\theta} \leq C\varphi(\delta),$$

while the function φ satisfies the conditions

- (4) φ is a non-negative continuous function on $(0, 1)$ and $\varphi \neq 0$;
- (5) For every δ_1, δ_2 such that $0 \leq \delta_1 \leq \delta_2 \leq 1$ holds $\varphi(\delta_1) \leq C_1 \varphi(\delta_2)$;
- (6) For every δ such that $0 \leq \delta \leq \frac{1}{2}$ holds $\varphi(2\delta) \leq C_2 \varphi(\delta)$,

where constants¹ C, C_1 and C_2 do not depend on δ_1, δ_2 and δ .

A more detailed approach to the classes $N(p, \theta, r, \lambda, \varphi)$ is given in [8] (see also [5, p. 298]). In our paper we give the necessary and sufficient condition in terms of monotone Fourier coefficients for a function $f \in L_p[0, 2\pi]$ to belong to the class $N(p, \theta, r, \lambda, \varphi)$.

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¹Without mentioning it explicitly, we will consider all the constants positive.

2. Now we formulate our results.

Theorem 2.1. *A function f belongs to the class $N(p, \theta, r, \lambda, \varphi)$ if and only if²*

$$\left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{(r+\lambda)\theta-1} \right)^{1/\theta} \leq C\varphi \left(\frac{1}{n} \right), \quad (2.1)$$

where constant C does not depend on n .

Theorem 2.2. *For a function $f \in L_p[0, 2\pi]$, $1 < p < \infty$, such that*

$$f(x) \sim \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu x, \quad a_{\nu} \downarrow 0, \quad (2.2)$$

to belong to the class $N(p, \theta, r, \lambda, \varphi)$ it is necessary and sufficient that its Fourier coefficients satisfy the condition

$$\left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{r\theta+\lambda\theta+\theta-\theta/p-1} \right)^{1/\theta} \leq C\varphi \left(\frac{1}{n} \right),$$

where constant C does not depend on n .

Remark 1. Put $\varphi(\delta) = \delta^{\alpha}$, $0 < \alpha < \lambda$, in the definition of the class $N(p, \theta, r, \lambda, \varphi)$, we obtain [8] the Nikol'skiĭ class $H_p^{r+\alpha}$. Thus Theorems 2.1 and 2.2 give the single coefficient condition

$$a_{\nu} \leq \frac{C}{\nu^{r+\alpha+1-\frac{1}{p}}},$$

for $f \in H_p^{r+\alpha}$, given in [7] (see also [3]), where the function f is given by (2.2).

Remark 2. If $\varphi(\delta) \geq C$, then we obtain [8] the Besov class $B_p^{\theta r}$. Thus Theorems 2.1 and 2.2 give the necessary and sufficient condition

$$\sum_{\nu=1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} < \infty$$

for $f \in B_p^{\theta r}$, given in [9] (see also [4]), where the function f is given by (2.2).

3. In order to establish our results, we use the following lemmas.

Lemma 3.1. *Let $0 < \alpha < \beta < \infty$ and $a_{\nu} \geq 0$. The following inequality holds true*

$$\left(\sum_{\nu=1}^n a_{\nu}^{\beta} \right)^{1/\beta} \leq \left(\sum_{\nu=1}^n a_{\nu}^{\alpha} \right)^{1/\alpha}.$$

Proof of the lemma is due to Jensen [6, p. 43].

Lemma 3.2. *Let $\{a_{\nu}\}_{\nu=1}^{\infty}$ be a sequence of non-negative numbers, $\alpha > 0$, λ a real number, m and n positive integers such that $m < n$. Then*

(1) *for $1 \leq p < \infty$ the following equalities hold*

$$\begin{aligned} \sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_{\nu} \nu^{\lambda} \right)^p &\leq C_1 \sum_{\mu=m}^n \mu^{\alpha-1} (a_{\mu} \mu^{\lambda+1})^p, \\ \sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^{\mu} a_{\nu} \nu^{\lambda} \right)^p &\leq C_2 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^p; \end{aligned}$$

²Here and below we assume that the parameters θ , r , λ and k satisfy the condition 2, and the function φ satisfies the conditions 4–6 of the definition of the class $N(p, \theta, r, \lambda, \varphi)$.

(2) for $0 < p \leq 1$ the following equalities hold

$$\sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p \geq C_3 \sum_{\mu=m}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^\mu a_\nu \nu^\lambda \right)^p \geq C_4 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_\nu\}_{\nu=1}^\infty$.

Proof of the lemma is given in [6, p. 308].

We write $a_\nu \downarrow$ if $\{a_\nu\}_{\nu=1}^\infty$ is a monotone-decreasing sequence of non-negative numbers, i.e. if $a_\nu \geq a_{\nu+1} \geq 0$ ($\nu = 1, 2, \dots$).

Lemma 3.3. Let $a_\nu \downarrow$, $\alpha > 0$, λ a real number, m and n positive integers. Then

(1) for $1 \leq p < \infty$, $n \geq 16m$ the following equalities hold

$$\sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p \geq C_1 \sum_{\mu=8m}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^\mu a_\nu \nu^\lambda \right)^p \geq C_2 \sum_{\mu=4m}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p;$$

(2) for $0 < p \leq 1$, $n \geq 4m$ the following equalities hold

$$\sum_{\mu=4m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p \leq C_3 \sum_{\mu=m}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$\sum_{\mu=4m}^n \mu^{-\alpha-1} \left(\sum_{\nu=4m}^\mu a_\nu \nu^\lambda \right)^p \leq C_4 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_\nu\}_{\nu=1}^\infty$.

Proof of the lemma is given in [2].

Lemma 3.4. Let $a_\nu \downarrow$, $\alpha > 0$, λ a real number, m and n positive integers. For $0 < p < \infty$ the following inequalities hold

$$C_1 \sum_{\mu=1}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p \leq \sum_{\mu=1}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p \leq C_2 \sum_{\mu=1}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

$$C_3 \sum_{\mu=1}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p \leq \sum_{\mu=1}^n \mu^{-\alpha-1} \left(\sum_{\nu=1}^\mu a_\nu \nu^\lambda \right)^p \leq C_4 \sum_{\mu=1}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p,$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_\nu\}_{\nu=1}^\infty$.

The lemma is also proved in [2].

Lemma 3.5. Let $f \in L_p[0, 2\pi]$ for a fixed p from the interval $1 < p < \infty$ and let

$$f(x) \sim \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad a_\nu \downarrow 0.$$

The following inequalities hold

$$\begin{aligned} C_1 \frac{1}{n^k} \left(\sum_{\nu=1}^n a_\nu^p \nu^{(k+1)p-2} \right)^{1/p} + \left(\sum_{\nu=n+1}^{\infty} a_\nu^p \nu^{p-2} \right)^{1/p} &\leq \omega_k \left(f, \frac{1}{n} \right)_p \\ &\leq C_2 \frac{1}{n^k} \left(\sum_{\nu=1}^n a_\nu^p \nu^{(k+1)p-2} \right)^{1/p} + \left(\sum_{\nu=n+1}^{\infty} a_\nu^p \nu^{p-2} \right)^{1/p}, \end{aligned}$$

where constants C_1 and C_2 do not depend on n and f .

The lemma is proved in [9].

4. Now we prove our results.

Proof of Theorem 2.1. Put

$$I_1 = \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^\theta dt, \quad I_2 = \int_{\frac{1}{n+1}}^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^\theta dt.$$

We have [6, p. 55]

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^\theta dt = \sum_{\nu=n+1}^{\infty} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} \omega_k(f, t)_p^\theta dt \\ &\leq \sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} dt \leq C_1 \sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} \end{aligned}$$

and, taking into account properties of modulus of smoothness [10, p. 116],

$$I_1 \geq \sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu+1} \right)_p^\theta \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} dt \geq C_2 \sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1}.$$

In an analogous way we estimate

$$I_2 \leq \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-(r+\lambda)\theta-1} dt \leq C_3 \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1}$$

and

$$I_2 \geq \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu+1} \right)_p^\theta \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-(r+\lambda)\theta-1} dt \geq C_4 \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1}.$$

Let $f \in N(p, \theta, r, \lambda, \varphi)$. For a positive integer n we put $\delta = \frac{1}{n+1}$. Then we have

$$\begin{aligned} I^\theta &= I_1 + \delta^{\lambda\theta} I_2 \\ &\geq C_5 \left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1} \right). \end{aligned}$$

Hence we obtain

$$\begin{aligned} J &= \left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1} \right)^{1/\theta} \\ &\leq C_6 I \leq C_7 \varphi(\delta) = C_7 \varphi \left(\frac{1}{n+1} \right) \leq C_8 \varphi \left(\frac{1}{n} \right), \end{aligned}$$

which proves inequality (2.1).

Now we suppose that inequality (2.1) holds. For $\delta \in (0, 1)$ we choose the positive integer n satisfying $\frac{1}{n+1} < \delta \leq \frac{1}{n}$. Then, taking into consideration the estimates from above for I_1 and I_2 we have

$$\begin{aligned} I^\theta &= \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^\theta dt + \int_{\frac{1}{n+1}}^\delta t^{-r\theta-1} \omega_k(f, t)_p^\theta dt \\ &\quad + \delta^\lambda \int_\delta^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^\theta dt \leq I_1 + \delta^\lambda I_2 \\ &\leq C_9 \left(\sum_{\nu=n+1}^\infty \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1} \right). \end{aligned}$$

Whence

$$I \leq C_{10} J \leq C_{11} \varphi \left(\frac{1}{n} \right) \leq C_{12} \varphi \left(\frac{1}{2n} \right) \leq C_{13} \varphi(\delta),$$

implying $f \in N(p, \theta, r, \lambda, \varphi)$.

Proof of Theorem 2.1 is completed. \square

Proof of Theorem 2.2. Theorem 2.1 implies that the condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$\sum_{\nu=n+1}^\infty \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1} \leq C_1 \varphi \left(\frac{1}{n} \right)^\theta,$$

where constant C_1 does not depend on n . Lemma 3.5 yields that the last estimate is equivalent to the estimate [1, p. 31]

$$\begin{aligned} &\sum_{\nu=n+1}^\infty \nu^{(r-k)\theta-1} \left(\sum_{\mu=1}^\nu a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p} + \sum_{\nu=n+1}^\infty \nu^{r\theta-1} \left(\sum_{\mu=\nu}^\infty a_\mu^p \mu^{p-2} \right)^{\theta/p} \\ &\quad + n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{(r+\lambda-k)\theta-1} \left(\sum_{\mu=1}^\nu a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\quad + n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^\infty a_\mu^p \mu^{p-2} \right)^{\theta/p} \leq C_2 \varphi \left(\frac{1}{n} \right)^\theta, \end{aligned}$$

where constant C_2 does not depend on n . Hence, if we denote the terms on the left-hand side of the inequality by J_1 , J_2 , J_3 and J_4 respectively, then condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$J_1 + J_2 + J_3 + J_4 \leq C_2 \varphi \left(\frac{1}{n} \right)^\theta. \quad (4.1)$$

Now we estimate the terms J_1 , J_2 , J_3 and J_4 from below and above by means of expression taking part in the condition of the theorem.

First we estimate J_1 and J_2 from below. We have

$$\begin{aligned} J_1 &= \sum_{\nu=n+1}^\infty \nu^{(r-k)\theta-1} \left(\sum_{\mu=1}^\nu a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\geq \sum_{\nu=n+1}^\infty \nu^{-(k-r)\theta-1} \left(\sum_{\mu=n+1}^\nu a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p}. \end{aligned}$$

For $k - r > 0$, making use of Lemmas 3.2 and 3.3 we obtain

$$\begin{aligned} J_1 &\geq C_3 \sum_{\nu=4(n+1)}^{\infty} \nu^{-(k-r)\theta-1} (a_\nu^p \nu^{(k+1)p-2\nu})^{\theta/p} \\ &= C_3 \sum_{\nu=4(n+1)}^{\infty} a_\nu^\theta \nu^{r\theta+\theta-\theta/p-1}. \end{aligned} \quad (4.2)$$

In an analogous way, for $r\theta > 0$ we get

$$J_2 = \sum_{\nu=n+1}^{\infty} \nu^{r\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_\mu^p \mu^{p-2} \right)^{\theta/p} \geq C_4 \sum_{\nu=8(n+1)}^{\infty} a_\nu^\theta \nu^{r\theta+\theta-\theta/p-1}. \quad (4.3)$$

We estimate the term J_2 from above:

$$J_2 \leq C_5 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} \nu^{r\theta-1} (a_\nu^p \nu^{p-2\nu})^{\theta/p} = C_5 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_\nu^\theta \nu^{r\theta+\theta-\theta/p-1}. \quad (4.4)$$

For J_1 we have

$$\begin{aligned} J_1 &\leq C_6 \left(\sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left(\sum_{\mu=n+1}^{\nu} a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p} \right. \\ &\quad \left. + \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left(\sum_{\mu=1}^n a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p} \right), \end{aligned}$$

and applying once more Lemmas 3.2 and 3.3 we obtain

$$J_1 \leq C_7 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_\nu^\theta \nu^{r\theta+\theta-\theta/p-1} + n^{-(k-r)\theta} \left(\sum_{\mu=1}^n a_\mu^p \mu^{(k+1)p-2} \right)^{\theta/p}. \quad (4.5)$$

Put

$$I_1 = n^{-(k-r)\theta} \sum_{\mu=1}^n a_\mu^p \mu^{(k+1)p-2}.$$

Then for

$$I_2 = I_1 n^{(k-r)\theta},$$

taking into account that $(k+1)p-2 \geq 0$ and $a_\nu \downarrow 0$ we get

$$\begin{aligned} I_2 &= \sum_{\mu=1}^n a_\mu^p \mu^{(k+1)p-2} \leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_\mu^p \mu^{(k+1)p-2} + a_{\lceil \frac{n}{2} \rceil+1}^p \sum_{\mu=\lceil \frac{n}{2} \rceil+1}^n \mu^{(k+1)p-2} \\ &\leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_\mu^p \mu^{(k+1)p-2} + C_8 n^{(k+1)p-1} a_{\lceil \frac{n}{2} \rceil+1}^p \leq C_9 \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_\mu^p \mu^{(k+1)p-2}. \end{aligned}$$

Since $k - r - \lambda > 0$, we have

$$\begin{aligned} I_1^{\theta/p} &\leq C_{10} n^{-(k-r)\theta} \left(\sum_{\mu=1}^{\lfloor \frac{n}{2} \rfloor} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\leq C_{11} n^{-\lambda\theta} \sum_{\nu=\lfloor \frac{n}{2} \rfloor}^n \nu^{-(k-r-\lambda)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\leq C_{11} n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{-(k-r-\lambda)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}. \end{aligned}$$

Applying Lemma 3.4 we obtain

$$\begin{aligned} I_1^{\theta/p} &\leq C_{12} n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{-(k-r-\lambda)\theta-1} (a_{\nu}^p \nu^{(k+1)p-2} \nu)^{\theta/p} \\ &= C_{12} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \end{aligned}$$

From (4.5) it follows that

$$J_1 \leq C_{13} \left(\sum_{\nu=\lfloor \frac{n+1}{4} \rfloor}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \quad (4.6)$$

This way, inequalities (4.2), (4.3), (4.4) and (4.6) yield

$$\begin{aligned} C_{14} \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} &\leq J_1 + J_2 \\ &\leq C_{15} \left(\sum_{\nu=\lfloor \frac{n+1}{4} \rfloor}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (4.7)$$

Now we estimate J_3 and J_4 . Put

$$A_1 = n^{\lambda\theta} J_3 = \sum_{\nu=1}^n \nu^{(r+\lambda-k)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}$$

and

$$A_2 = n^{\lambda\theta} J_4 = \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p},$$

applying Lemma 3.4 for $r + \lambda - k < 0$ we get

$$A_1 \leq C_{16} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \quad (4.8)$$

We estimate A_2 in an analogous way:

$$\begin{aligned} A_2 &\leq C_{17} \left(\sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^n a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right. \\ &\quad \left. + \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right) \\ &\leq C_{18} \left(\sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + n^{(r+\lambda)\theta} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right). \end{aligned} \quad (4.9)$$

We estimate the series

$$B = \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p}.$$

First let $\frac{\theta}{p} > 1$. Applying Hölder inequality we have

$$\begin{aligned} \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} &\leq \left(\sum_{\mu=n+1}^{\infty} (a_{\mu}^p \mu^{p-1+rp-p/\theta})^{\theta/p} \right)^{p/\theta} \\ &\quad \times \left(\sum_{\mu=n+1}^{\infty} (\mu^{-(rp-p/\theta+1)\theta/(\theta-p)})^{(\theta-p)/\theta} \right). \end{aligned}$$

Since $(rp - \frac{p}{\theta} + 1)\frac{\theta}{\theta-p} = rp\frac{\theta}{\theta-p} + 1 > 1$, we get

$$\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \leq C_{19} n^{-rp} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \mu^{\theta-\theta/p+rp-1} \right)^{p/\theta}.$$

So, for $\frac{\theta}{p} > 1$ we have proved that

$$B \leq C_{20} n^{-r\theta} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \mu^{r\theta+\theta-\theta/p-1}.$$

Let $\frac{\theta}{p} \leq 1$. For given n we choose the positive integer N such that $2^N \leq n+1 < 2^{N+1}$. Then we have

$$\begin{aligned} B &\leq \left(\sum_{\mu=2^N}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \leq \left(\sum_{\nu=N}^{\infty} a_{2^{\nu}}^p \sum_{\mu=2^{\nu}}^{2^{\nu+1}-1} \mu^{p-2} \right)^{\theta/p} \\ &\leq C_{21} \left(\sum_{\nu=N}^{\infty} a_{2^{\nu}}^p 2^{\nu(p-1)} \right)^{\theta/p}. \end{aligned}$$

Making use of Lemma 3.1 we obtain

$$\begin{aligned} B &\leq C_{21} \sum_{\nu=N}^{\infty} a_{2^{\nu}}^{\theta} 2^{\nu(\theta-\theta/p)} \leq C_{22} \sum_{\nu=N}^{\infty} \sum_{\mu=2^{\nu-1}}^{2^{\nu}-1} a_{\mu}^{\theta} \mu^{\theta-\theta/p-1} \\ &= C_{22} \sum_{\nu=2^{N-1}}^{\infty} a_{\nu}^{\theta} \nu^{\theta-\theta/p-1} \leq C_{22} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{\theta-\theta/p-1} \\ &\leq C_{22} \left[\frac{n+1}{4} \right]^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}. \end{aligned}$$

Since for $n \geq 3$ holds $\lceil \frac{n+1}{4} \rceil \geq \frac{n}{12}$, we get

$$B \leq C_{23} n^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}.$$

This way, for $0 < \frac{\theta}{p} < \infty$ we proved that

$$B \leq C_{24} n^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}.$$

Hence (4.9) yields

$$A_2 \leq C_{25} \left(\sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + n^{\lambda\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right).$$

Now, from (4.8) it follows that

$$\begin{aligned} J_3 + J_4 &= n^{-\lambda\theta} (A_1 + A_2) \\ &\leq C_{26} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (4.10)$$

Further, we estimate the series

$$A_3 = \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} = A_4 + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1},$$

where is

$$\begin{aligned} A_4 &= \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^n a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \leq C_{27} a_{\lceil \frac{n+1}{4} \rceil}^{\theta} n^{r\theta+\theta-\theta/p} \\ &\leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^{\lceil \frac{n+1}{4} \rceil} a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \end{aligned}$$

Whence

$$A_3 \leq C_{29} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \quad (4.11)$$

Making use of (4.11) and (4.10) we have

$$J_3 + J_4 \leq C_{30} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right).$$

Hence, applying (4.11) in (4.7) we obtain

$$\begin{aligned} J_1 + J_2 + J_3 + J_4 &\leq C_{31} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (4.12)$$

Now we estimate A_1 and A_2 from below. Making use of Lemma 3.4 we get

$$A_1 \geq C_{32} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1},$$

and in an analogous way

$$A_2 \geq \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^n a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \geq C_{33} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

Hence

$$A_1 + A_2 \geq C_{34} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

This way the following inequality holds

$$J_3 + J_4 \geq C_{35} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

From (4.7) it follows that

$$\begin{aligned} J_1 + J_2 + J_3 + J_4 \\ \geq C_{36} \left(\sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (4.13)$$

Since

$$\begin{aligned} \sum_{\nu=n+1}^{\nu=8(n+1)-1} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} &\leq C_{37} a_n^{\theta} n^{r\theta+\theta-\theta/p} \\ &\leq C_{38} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \end{aligned}$$

holds, we have

$$\begin{aligned} \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \\ \leq C_{39} \left(\sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned}$$

Now, estimates (4.13) and (4.12) imply

$$\begin{aligned} C_{40} \left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right) \\ \leq J_1 + J_2 + J_3 + J_4 \\ \leq C_{41} \left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned}$$

This way we proved that condition (2.1) is equivalent to the condition of the theorem. Since condition (2.1) is equivalent to the condition $f \in N(p, \theta, r, \lambda, \varphi)$, proof of Theorem 2.2 is completed. \square

REFERENCES

1. N. K. Bari, *Trigonometricheskie ryady*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. MR 23 #A3411
2. F. M. Berisha, *On some weighted l_p type inequalities about monotone sequences*, (preprint).
3. M. Q. Berisha, *O koefitsientakh Fur'e nekotorykh klassov funktsii*, Glas. Mat. Ser. III **16(36)** (1981), no. 1, 75–90. MR 83a:42004
4. ———, *O koefitsientakh Fur'e funktsii prinallezhashchikh klassam Besova tipa $B(p, \theta, \alpha)$* , Serdica **11** (1985), no. 1, 79–85. MR 87a:42010
5. O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, *Integral'nye predstavleniya funktsii i teoremy vlozheniya*, Fizmatlit "Nauka", Moscow, 1996. MR 98b:46037
6. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1988, (Russian translation, Gosudarstv. Izdat. Inostrannoĭ Literatury, Moscow, 1948). MR 89d:26016
7. A. A. Konyushkov, *O klassakh lipshitsa*, Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), no. 3, 423–448.
8. B. Laković, *Ob odnom klasse funktsii*, Mat. Vesnik **39** (1987), no. 4, 405–415. MR 89h:41062
9. M. K. Potapov and M. Q. Berisha, *Moduli gladkosti i koefitsienty Fur'e periodicheskikh funktsii ednogo peremennogo*, Publ. Inst. Math. (Beograd) (N.S.) **26(40)** (1979), 215–228. MR 81e:42009
10. A. F. Timan, *Teoriya priblizheniya funktsii deĭstvitel'nogo peremennogo*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. MR 22 #8257

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